Section 7 – Fundamentals of Sequences and Series

7.1 Definition and examples of sequences

A sequence can be thought of as an infinite list of numbers.

Example:

(i) 0, -5, -10, -15, -20 ....

(ii) 1, 2, 3, 5, 7, 11, ….

(iii) $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \ldots\ldots$

Definition: A **sequence** is a function which has as its domain the set of positive integers.

The elements of the range of a sequence are called the **terms** of the sequence.

Consider the sequence $f(1) = a_1$, $f(2) = a_2$, $f(3) = a_3\ldots$
We denote this sequence by a_1, a_2, a_3, …… or by $\{a_k\}$ and the **general term** or the ***n*th term** of the sequence by a_n.

Example:

(i) Write down the first 3 terms of the sequence in which the general term is $a_n = n^2$.

(ii) Write down the general term of the sequence 1, 3, 6, 10, 15, 21, …..

Solution:

(i) $a_1 = 1$, $a_2 = 4$, $a_3 = 9$

(ii) $a_n = \frac{n(n+1)}{2}$

Consider the sequence 0, 5, 10, 15, 20, …..
In this sequence, each succeeding term is larger than the preceding term. Such sequences are called increasing sequences.

A sequence a_1, a_2, a_3, …. is said to be **increasing** if $a_n < a_{n+1}$ for all $n =$ 1, 2, 3, ……

Consider the sequence $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \ldots\ldots$
In this sequence, each succeeding term is smaller than the preceding term. Such sequences are called decreasing sequences.

A sequence a_1, a_2, a_3, …. is said to be **decreasing** if $a_n > a_{n+1}$ for all $n =$ 1, 2, 3, ……

7.2 Series and the sequence of terms of a series

Let $\{a_k\}$ be a sequence. A **series,** denoted by $\sum_{k=1}^{\infty} a_k$ is defined to be the sequence $\{S_n\}$, where $S_n = a_1 + a_2 + ... + a_n$. The numbers a_k are called the **terms** of the series, and the numbers S_n are called the **partial sums** of the series.

A series of which the general term is known can be written in a compact form using the **summation notation** or **sigma notation**. We use the capital Greek letter sigma (Σ) to denote the sum.
The sum S_n of the first n terms of a sequence with general term a_i is denoted using the sigma notation by

$$S_n = \sum_{i=1}^{n} a_i$$

The letter i in the above is called the **index of summation**. The letter i indicates that we start adding from i = 1 and end with i = n. The letter i in the above summation is a dummy variable and can be replaced by any other letter.

Example:

(i) $$\sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2 = 1 + 4 + 9 + 16 = 30$$

(ii) $$\sum_{j=2}^{4} \frac{2j}{j+1} = \frac{2\times 2}{2+1} + \frac{2\times 3}{3+1} + \frac{2\times 4}{4+1} = \frac{4}{3} + \frac{6}{4} + \frac{8}{5} = \frac{4}{3} + \frac{3}{2} + \frac{8}{5} = \frac{40+45+48}{30} = \frac{133}{30} = 4\frac{13}{30}$$

(iii) $$\sum_{i=1}^{\infty} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + ...$$

7.3 Arithmetic Progressions

An **arithmetic progression** is a sequence of numbers such that any two successive terms differ by the same amount, called the **common difference** and denoted by d.

Example:

(i) 2, 5, 8, 11, 14, … (d = 3)
(ii) 15, 10, 5, 0, -5, …. (d = -5)

7.3.1 Formula for the n^{th} term

Suppose we consider the arithmetic progression 3, 6, 9, 12, 15,….

The common difference $d = 3$

$a_1 = 3$
$a_2 = 6 = 3 + (2 – 1) \times 3 = a_1 + (2 – 1) \times d$
$a_3 = 9 = 3 + (3 – 1) \times 3 = a_1 + (3 – 1) \times d$
$a_4 = 12 = 3 + (4 – 1) \times 3 = a_1 + (4 – 1) \times d$
Therefore we see that $a_n = a_1 + (n – 1) \times d$

This is true for any arithmetic progression $a_1, a_2, a_3, \ldots.$

$a_2 = a_1 + d = a_1 + (2 – 1)d$
$a_3 = a_2 + d = (a_1 + d) + d = a_1 + 2d = a_1 + (3 – 1)d$
$a_4 = a_3 + d = (a_1 + 2d) + d = a_1 + 3d = a_1 + (4 – 1)d$
Proceeding in this manner we obtain
$a_n = a_1 + (n – 1) \times d$

The $\boldsymbol{n}^{\text{th}}$ **term** of an **arithmetic progression** is given by the formula $\boldsymbol{a_n = a_1 + (n – 1) \times d}$ where a_1 is the first term and d is the common difference

Example:

(i) Find the 12th term of the arithmetic progression in which the first term is 12 and the common difference is -3.

(ii) Find the first 4 terms of the arithmetic progression in which the first term is 8 and the common difference is 4.

(iii) Find the first 3 terms of the arithmetic progression in which the 10th term is 10 and the common difference is 2.

(iv) If $a_n = 20$, $a_1 = 5$ and $d = 3$, how much is n?

(v) If $a_4 = -2$ and $a_8 = 18$, what is a_{12}?

(vi) How many numbers are there between 20 and 70 which are divisible by 3?

Solution:

(i) $a_{12} = a_1 + (12-1)\times(-3) = 12 + (11\times -3) = 12 - 33 = -21$.

(ii) $a_1 = 8$, $a_2 = 12$, $a_3 = 16$, $a_4 = 20$.

(iii) $a_{10} = a_1 + (10-1)\times 2$

$10 = a_1 + 18$

$a_1 = -8$

Therefore, $a_1 = -8$, $a_2 = -6$, $a_3 = -4$.

(iv) $a_n = a_1 + (n-1)\times d$

$20 = 5 + (n - 1) \times 3$

$15 = (n - 1) \times 3$

$n - 1 = 5$

$n = 6$

(v) $a_4 = a_1 + (4-1)\times d$ and $a_8 = a_1 + (8-1)\times d$

$-2 = a_1 + 3d$ ----------------- (a)

$18 = a_1 + 7d$ ----------------- (b)

Subtracting equation (a) from equation (b)

$20 = 4d$

$d = 5$

Substituting into (a);

$-2 = a_1 + (3\times 5)$

$a_1 = -2 - 15 = -17$

Therefore, $a_{12} = a_1 + 11d = -17 + (11\times 5) = -17 + 55 = 38$

(vi) The first number divisible by 3 between 20 and 70 is 21. Therefore $a_1 = 21$. The last number divisible by 3 between 20 and 70 is 69. Let $a_n = 69$. Then we need to find n when $a_1 = 21$, $d = 3$ and $a_n = 69$.

$a_n = a_1 + (n-1)\times d$

$69 = 21 + 3(n - 1)$

$48 = 3(n - 1)$

$16 = n - 1$

$n = 17$.

Thus there are 17 numbers between 20 and 70 which are divisible by 3.

7.3.2 <u>Formulae for the sum of the first *n* terms</u>

Consider the sum of the first n terms $a_1, a_2, a_3, \ldots\ldots a_{n-1}, a_n$ of an arithmetic progression.

$S_n = a_1 + a_2 + \ldots\ldots + a_n$

$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \ldots\ldots + (a_1 + (n-1)d)$ -------------(i)

This sum may also be written in the reverse order

$S_n = a_n + (a_n - d) + (a_n - 2d) + \ldots\ldots + (a_n - (n-1)d)$ ---------------(ii)

Adding (i) and (ii) we obtain

$2S_n = n(a_1 + a_n)$

Therefore $S_n = \frac{n}{2}(a_1 + a_n)$

Since $a_n = a_1 + (n-1) \times d$,

$S_n = \frac{n}{2}(a_1 + a_n) = \frac{n}{2}\{a_1 + a_1 + (n-1)d\} = \frac{n}{2}\{2a_1 + (n-1)d\}$

> The **sum of the first *n* terms** of an arithmetic progression in which the first term is a_1, common difference is d and n^{th} term is a_n is given by
>
> $$S_n = \frac{n}{2}(a_1 + a_n) \text{ or } S_n = \frac{n}{2}\{2a_1 + (n-1)d\}$$

Example:

(i) Find the sum of the first 16 terms of the arithmetic progression 4, 8, 12, 16, 20, …..

(ii) Find the sum of the numbers which are divisible by 6 that lie between 50 and 100.

(iii) The 5^{th} term of an arithmetic progression is 14 and the 11^{th} term is 26. Find the sum of the first 15 terms of the progression.

(iv) An auditorium hall has 40 chairs in the first row. Each successive row has two chairs more than the previous row. How many chairs are there in total in the first 20 rows? How many of the front rows would 376 people occupy?

Solution:

(i) $S_n = \frac{n}{2}\{2a_1 + (n-1)d\}$

$S_{16} = \frac{16}{2}\{2 \times 4 + (16-1) \times 4\} = 8(8 + 60) = 8 \times 68 = 544$.

(ii) $a_1 = 54$, $d = 6$, $a_n = 96$.

$a_n = a_1 + (n-1) \times d$

$96 = 54 + 6(n-1)$

$42 = 6(n-1)$

$n - 1 = 7$

$n = 8$

$S_n = \frac{n}{2}(a_1 + a_n)$; $S_8 = \frac{8}{2}(54 + 96) = 4 \times 150 = 600$.

(iii) $a_5 = 14$, $a_{11} = 26$.

$14 = a_1 + 4d$ ----------------(a)

$26 = a_1 + 10d$ -------------------(b)

Subtracting equation (a) from equation (b) we obtain

$12 = 6d$

$d = 2$.

From (a) we obtain $a_1 = 14 – 8 = 6$

Therefore, $S_{15} = \frac{15}{2}\{2\times 6+(15-1)\times 2\} = \frac{15}{2}\times(12+28) = \frac{15}{2}\times 40 = 300$.

(iv) $S_n = \frac{n}{2}\{2a_1 + (n-1)\times d\}$

$S_{20} = \frac{20}{2}\{(2\times 40)+(19\times 2)\} = 10(80+38) = 1180$

Thus the first 20 rows have 1180 chairs.

$S_n = 376$

Therefore

$376 = \frac{n}{2}\{2\times 40 + (n-1)\times 2\}$

$376 = n(40+n-1)$

$n^2 + 39n – 376 = 0$

$(n – 8)(n + 47) = 0$

$n = 8$ or $n = -47$

Therefore the number of front rows that 376 people would occupy is 8.

7.4 Geometric Progressions

A **geometric progression** is a sequence of numbers such that the ratio of any term to the preceding term is a fixed number called the **common ratio** and denoted by r.

Example:

(i) 2, 4, 8, 16, …..

(ii) 5, 15, 45, 135, ….

7.4.1 The formula for the n^{th} term

Consider a general geometric progression a_1, a_2, a_3, …… a_{n-1}, a_n, ….. with common ratio r.

Then $\frac{a_n}{a_{n-1}} = r$ for all $n = 2, 3, 4,$ …..

Therefore

$a_2 = a_1 r$

$a_3 = a_2 r = (a_1 r)r = a_1 r^2$

$a_4 = a_3r = (a_1r^2)r = a_1r^3$
Proceeding in this manner we obtain
$a_n = a_1r^{n-1}$

The $\boldsymbol{n^{th}}$ **term** of a **geometric progression** is given by the formula $\boldsymbol{a_n = a_1r^{n-1}}$ where a_1 is the first term of the progression and r is the common ratio

Example:

(i) Find the 7th term of the geometric progression $1, \frac{1}{3}, \frac{1}{3^2}, ...$

(ii) Find the first two terms of the geometric progression in which the common ratio is 3 and the 6th term is 162

Solution:

(i) $r = \frac{1}{3}$ and $a_1 = 1$. Therefore, $a_7 = 1(\frac{1}{3})^6 = \frac{1}{729}$

(ii) $a_6 = a_1r^5$. Therefore, $a_1 = \frac{162}{3^5} = \frac{162}{243} = \frac{2}{3}$ and $a_2 = 2$.

7.4.2 The formula for the sum of the first *n* terms

Consider the sum of the first n terms $a_1, a_2, a_3, \ldots\ldots a_{n-1}, a_n$ of a geometric progression.

$S_n = a_1 + a_1r + a_1r^2 + a_1r^3 + \ldots.. + a_1r^{n-1}$ ------------------ (1)

$r\,S_n = a_1r + a_1r^2 + a_1r^3 + a_1r^4 + \ldots.. + a_1r^n$ ------------------(2)

Subtracting equation (2) from equation (1) we obtain

$(1 - r)S_n = a_1 - a_1r^n$
$\qquad\qquad = a_1(1 - r^n)$

Therefore, $S_n = \frac{a_1(1-r^n)}{1-r}$ provided $r \neq 1$.

The sum of the first n terms of a geometric progression in which the first term is a_1 and common ratio is r is given by $$S_n = \frac{a_1(1-r^n)}{1-r}$$ provided $r \neq 1$.

Example:

(i) Amal starts working for an annual salary of Rs. 150,000. He is promised a 10% increment each year. What will be his salary in the 5th year? What would be his total earnings during the first 5 years?

(ii) Dileni decides to save money by making monthly deposits starting with an initial deposit of Rs. 100 and then each month doubling the amount she deposited the previous month. How much would she have saved at the end of 6 months?

Solution:

(i) $a_5 = a_1 r^4 = 150{,}000\left(\frac{110}{100}\right)^4 = 150{,}000\left(\frac{11}{10}\right)^4 = 219{,}615$; i.e., Amal's annual salary in the 5th year is Rs. 219,615

$$S_5 = \frac{150{,}000\left(1-\left(\frac{110}{100}\right)^5\right)}{1-\frac{110}{100}} = \frac{15(100000-161051)}{-1} = 915{,}765$$

i.e., Amal's total earnings during the first 5 years is Rs. 915,765

(ii) $S_6 = \frac{100(1-2^6)}{1-2} = 6300$; i.e., Dileni would have saved Rs. 6,300 at the end of 6 months.

7.5 The sum to infinity of a series and the convergence and divergence of series

Consider the series 1 + 2 + 4 + 8 + …..

The sum of the first n terms of this series is $S_n = \sum_{i=0}^{n} 2^i$

We see that this sum gets larger and larger as n increases. In such a case we say that S_n tends to infinity as n tends to infinity or that the series is **divergent**. We denote this by $S_n \to \infty$ as $n \to \infty$.

Now consider the series $1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+......$

The sum of the first n terms of this series is given by

$$S_n = \frac{a_1(1-r^n)}{1-r} = \frac{1(1-\frac{1}{2^n})}{1-\frac{1}{2}} = 2-\frac{1}{2^{n-1}}$$

We see that as n increases $\frac{1}{2^{n-1}}$ decreases and the partial sum S_n approaches 2.

In this case we say that the infinite series $\sum_{i=1}^{\infty} \frac{1}{2^{i-1}}$ is **convergent** and its sum to infinity is 2 or that S_n **converges** (to 2) as n tends to infinity.

Thus, if the sum of an infinite series is a finite number we say that the series is **convergent** (or that the series converges). If not we say that the series is **divergent** (or that the series diverges).

Consider the general geometric series $a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \ldots$..

The sum of the first n terms is

$$S_n = \frac{a_1(1-r^n)}{1-r} = \frac{a_1}{1-r} - \left(\frac{a_1}{1-r}\right) r^n \qquad r \neq 1$$

If $|r| < 1$, then as n increases to infinity, r^n decreases to zero.

Therefore S_n approaches $\frac{a_1}{1-r}$ as n approaches infinity when $|r| < 1$.

If $|r| \geq 1$, then S_n diverges as n approaches infinity.

The sum of the terms of an infinite geometric series in which the first term is a_1 and common ratio is r is given by

$$S = \sum_{k=0}^{\infty} a_1 r^k = \frac{a_1}{1-r}$$

provided $-1 < r < 1$.

Properties:

(i) If $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are convergent series and c is a constant, then $\sum_{k=1}^{\infty} ca_k$ and $\sum_{k=1}^{\infty} (a_k + b_k)$ are convergent series and

(a) $\sum_{k=1}^{\infty} ca_k = c \sum_{k=1}^{\infty} a_k$

(b) $\sum_{k=1}^{\infty} (a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$

(ii) If an infinite series is convergent, then the series obtained from this series by adding a finite number of terms or subtracting a finite number of terms is also convergent.

i.e., If a_1, a_2, a_3, …… is a sequence, then the series $\sum_{k=n}^{\infty} a_k$ converges if and only if the series $\sum_{k=m}^{\infty} a_k$ converges (here m, n are natural numbers).

<u>Example</u>:

(i) Find the sum of the series $1-\frac{3}{4}+\frac{9}{16}-\frac{27}{64}+.....$

(ii) Determine whether the following series converge or diverge. If they converge, find their sum.

(a) $\sum_{k=1}^{\infty} 4\left(\frac{3^k+4^k}{5^k}\right)$

(b) $\sum_{k=1}^{\infty} \frac{3^{k+2}}{7^{k-1}}$

(c) $\sum_{k=3}^{\infty} \frac{4^{k+2}}{3^{k-1}}$

(d) $\sum_{k=3}^{\infty} \frac{5\times 3^{k+2}}{5^k}$

<u>Solution</u>:

(i) $1-\frac{3}{4}+\frac{9}{16}-\frac{27}{64}+.....$

This is a geometric series with $a_1 = 1$ and $r = -\frac{3}{4}$. Since $-1 < r < 1$, the series is convergent

$$1-\frac{3}{4}+\frac{9}{16}-\frac{27}{64}+..... = \frac{1}{1-\left(-\frac{3}{4}\right)} = \frac{1}{1+\frac{3}{4}} = \frac{4}{7}$$

(ii) (a) $\sum_{k=1}^{\infty} 4\left(\frac{3^k+4^k}{5^k}\right) = 4\sum_{k=1}^{\infty}\frac{3^k}{5^k} + 4\sum_{k=1}^{\infty}\frac{4^k}{5^k}$ by the above properties since $\left|\frac{3}{5}\right| < 1, \left|\frac{4}{5}\right| < 1$

$$= 4\left(\frac{3}{5}\right)\sum_{k=1}^{\infty}\left(\frac{3}{5}\right)^{k-1} + 4\left(\frac{4}{5}\right)\sum_{k=1}^{\infty}\left(\frac{4}{5}\right)^{k-1}$$

$$= \frac{12}{5}\left(\frac{1}{1-\frac{3}{5}}\right)+\frac{16}{5}\left(\frac{1}{1-\frac{4}{5}}\right)=\frac{12}{2}+16=22$$

(b) $\displaystyle\sum_{k=1}^{\infty}\frac{3^{k+2}}{7^{k-1}} = 3^3\sum_{k=1}^{\infty}\left(\frac{3}{7}\right)^{k-1}=27\left(\frac{1}{1-\frac{3}{7}}\right)=\frac{27\times 7}{4}=\frac{189}{4}$

(c) $\displaystyle\sum_{k=3}^{\infty}\frac{4^{k+2}}{3^{k-1}} = 4^3\sum_{k=3}^{\infty}\frac{4^{k-1}}{3^{k-1}}=4^3\sum_{k=1}^{\infty}\left(\frac{4}{3}\right)^2\left(\frac{4}{3}\right)^{k-1}$. Since $\frac{4}{3}>1$ the series diverges

(d) $\displaystyle\sum_{k=3}^{\infty}\frac{5\times 3^{k+2}}{5^k} = \left(\frac{3^5}{5^2}\right)\sum_{k=1}^{\infty}\frac{3^{k-1}}{5^{k-1}}=\frac{3^5}{25}\left(\frac{1}{1-\frac{3}{5}}\right)=\frac{243}{10}$