Section 5: Ratios and Proportions

5.1 Ratios

There are several ways of comparing the sizes of similar quantities.

Example:

A bar of chocolates was divided into 12 pieces. Ramani ate 4 pieces and Dilan ate the remaining 8 pieces.
We can compare the number of pieces of chocolate that they ate in various ways.

(i) The difference between the pieces of chocolate they ate is $8 - 4 = 4$. Therefore, Ramani ate 4 pieces less than Dilan did.

(ii) $\dfrac{\text{The number of pieces of chocolate that Ramani ate}}{\text{The number of pieces of chocolate that Dilan ate}} = \dfrac{4}{8} = \dfrac{1}{2}$.
i.e., Ramani ate half the number of pieces that Dilan did.

A comparison such as (ii) of the above example is known as a comparison by division. When two similar quantities are compared by division, a ratio is formed. For the above example we say that the ratio of the number of pieces of chocolate that Ramani ate to the number of pieces of chocolate that Dilan ate is 4 to 8.

A **ratio** between two quantities is a number that expresses the numerical relationship between the two quantities, when the two quantities are compared by division. The ratio of a to b ($b \neq 0$) is denoted by $\dfrac{a}{b}$ or a to b or $a : b$. a is the 'first term' and is known as the **antecedent** and 'b' is the second term or **consequent**.

In the above example, the ratio 4:8 can also be written as 1:2, since $\dfrac{4}{8} = \dfrac{1}{2}$. 1:2 expresses the ratio in the **lowest terms** or in the **simplest form**.

Since $\dfrac{a}{b} = \dfrac{ma}{mb}$ for any natural number m, the ratio $a : b$ equals the ratio $ma : mb$;
i.e., *the value of a ratio remains the same if the antecedent and the consequent are* ***multiplied or divided by the same quantity***.

Example:

(i) Tashya is 12 years old and her mother is 36 years old. What is the ratio between her age and her mother's age? What was the ratio between her age and her mother's age 4 years ago?

(ii) The ratio of the lengths of two pieces of cloth is 4:7. If the length of the shorter piece is 32cm, what is the length of the longer piece?

Solution:

(i) The ratio between Tashya's age and her mother's age is 12:36 or 1:3. Four years ago, Tashya was 8 years, while her mother was 32 years. Therefore the ratio between her age and her mother's age four years ago was 8:32 or 1:4.

(ii) $\frac{4}{7} = \frac{32}{x}$. Therefore, $x = \frac{32 \times 7}{4} = 56$; i.e., the length of the longer piece is 56 cm.

Ratio of Two Quantities

The magnitudes of two quantities of the same kind, such as two lengths, two weights, two areas etc., can be compared by means of a ratio. To do this, the measurements of the two quantities should be expressed in terms of the **same unit**. It is generally easier to convert to the smaller unit. Note that a ratio expressed in the form $\frac{a}{b}$ is always a number and is not expressed in terms of any particular unit.

Example: Find the ratio of the following

(i) 15cm to 2m
(ii) 300g to 2kg
(iii) 5 days to 3 weeks

Solution:

(i) 2m = 200cm. Therefore the ratio is 15 : 200 or 3 : 40
(ii) 2kg = 2000g. Therefore the ratio is 300 : 2000 or 3 : 20
(iii) 3 weeks = 21 days. Therefore the ratio is 5: 21

Division in a given Ratio

Let us now consider how a quantity can be divided into parts which bear given ratios to one another.

For example, if we say that a cake is shared among three people *A*, *B* and *C* in the ratio 2:3:3, it means that the cake is divided into 8 pieces (2 + 3 + 3) and *A* gets 2 pieces while *B* and *C* get three pieces each.

Example:

(i) Three friends decide to purchase a book that costs Rs. 2 250 by sharing the cost in the ratio 1:2:2. How much does the person who contributes the least pay?

(ii) The perimeter of a triangle is 36 cm. The sides of the triangle are in the ratio 1:2:3. What are the lengths of the sides?

(iii) The cost of a banana, a mango and a pineapple are in the ratio 1:3:4. If Sumana spent Rs. 32 for a pineapple, how much would she spend for a banana and a mango?

Solution:

(i) The sum of the terms of the ratio is 1 + 2 + 2 = 5. Thus the person who contributes the least pays $\frac{1}{5}^{\text{th}}$ of the cost, while the other two persons contribute $\frac{2}{5}^{\text{th}}$ of the cost each. So the amount that the person who contributes the least pays is Rs. $\frac{1}{5} \times 2\,250 =$ Rs. 450.

(ii) The sum of the terms of the ratio is 1 + 2 + 3 = 6. Therefore the shortest side is $\frac{1}{6}^{\text{th}}$ of 36 cm, the longest side is $\frac{3}{6}^{\text{th}}$ of 36 cm and the remaining side is $\frac{2}{6}^{\text{th}}$ of 36 cm. Thus the lengths of the sides of the triangle are $\frac{1}{6} \times 36$ cm = 6 cm, $\frac{1}{2} \times 36$ cm = 18 cm and $\frac{1}{3} \times 36$ cm = 12 cm respectively.

(iii) The price of a banana is $\frac{1}{4}$ th the price of a pineapple and the price of a mango is 3 times the price of a banana. Since the price of a pineapple is Rs. 32, the price of a banana is Rs. 8 and the price of a mango is Rs. 24.

Comparison of Ratios

To compare two ratios, they need to be first written in the simplest form and then their denominators (when written as fractions) have to be made equal.

Example:

In two examinations Shellomi scored 40 marks out of 50 marks and 70 marks out of 80 marks respectively. In which examination was her performance better?

Solution:

The ratio of the marks in the first examination is 40:50 or 4:5 or $\frac{4}{5}$.

The ratio of the marks in the second examination is 70:80 or 7:8 or $\frac{7}{8}$.

The least common multiple of the denominators 5 and 8 is 40.

Now $\frac{4}{5}=\frac{32}{40}$ and $\frac{7}{8}=\frac{35}{40}$. Thus Shellomi performed better in the second examination.

5.2 Proportions

A statement expressing the equality of two ratios is called a **proportion**. The four quantities compared are called the **terms** of the proportion. The first and last terms are called the **extremes** and the second and third terms are called the **means**.

Note:

1. Since every proportion is the equality of two ratios, it follows that the first and second terms must be of the same kind, and the third and fourth terms must also be of the same kind.
2. When four numbers are in proportion, the product of the extremes equals the product of the means since $\frac{a}{b}=\frac{c}{d} \Leftrightarrow ad = bc$.

5.2.1 Direct Proportion

Two quantities are said to be in direct proportion to each other (or to vary directly), if they both increase or decrease at the same rate. This relationship for two quantities S and T can be represented as $\frac{S_1}{S_2}=\frac{T_1}{T_2}$. This can also be written as $\frac{S_1}{T_1}=\frac{S_2}{T_2}=k$ or as $S = kT$ where k is a constant.

Example:

(i) The distance travelled by a vehicle moving with uniform velocity is directly proportional to the time taken.

(ii) The weight of an amount of water is directly proportional to its volume.

(iii) The circumference of a circle is directly proportional to its radius.

The following provide further examples.

Example:

(i) Suppose the distance between two cities measures 2.5 cm on a map. The scale on the map is 1 cm : 10 km. How far apart are the cities?

(ii) Two cities that are 15 km apart appear on a map 4 cm apart. What is the scale on the map?

(iii) If a person is paid a commission of Rs. 15 000 on the sale of a house worth Rs. 3 million, what would his commission be on the sale of a house worth Rs. 7.5 million, at the same rate?

Solution:

(i) Suppose the actual distance between the two cities is denoted by x km. Then $\frac{1}{2.5} = \frac{10}{x}$. Therefore $x = 2.5 \times 10 = 25$. Thus the distance between the two cities is 25 km.

(ii) Suppose the scale on the map is given by 1cm : x km. Then $\frac{1}{4} = \frac{x}{15}$. Therefore $x = \frac{15}{4} = 3.75$. Thus the scale is 1 cm : 3.75 km.

(iii) Suppose the person's commission on the sale of the house worth Rs. 7.5 million is Rs. x. Then $\frac{15000}{x} = \frac{3}{7.5}$. Therefore $x = \frac{15000 \times 7.5}{3} = 5000 \times 7.5$ $= 37500$. Thus his commission on the house would be Rs. 37 500/-.

5.2.2 Inverse Proportion

Two quantities are said to be in inverse proportion to each other if, when one quantity changes according to a certain ratio, the other quantity changes in the inverse ratio.

This relationship for two quantities S and T can be represented as $\frac{S_1}{S_2} = \frac{T_2}{T_1}$. This can also be written as $S_1T_1 = S_2T_2 = k$ or as $ST = k$ where k is a constant.

Example:

(i) When the distance travelled by a vehicle is fixed, then the speed of the vehicle is inversely proportional to the time taken. For example, if the speed is doubled, then the time taken to travel a fixed distance is halved.
(ii) When the temperature is constant, the volume of a fixed mass of gas is inversely proportional to the pressure on it.
(iii) The time taken to complete a fixed amount of work is inversely proportional to the number of persons involved in the work.

Example:

(i) Find the time taken for a vehicle travelling at a uniform speed of 55 km h^{-1} to travel a distance of 220 km.

(ii) If it takes 12 days for 10 men working 8 hours a day to complete a task, how many days will it take 15 men working 8 hours a day to complete the same task?

(iii) On a certain day, there are dry rations sufficient for 3 weeks for 10 students at a hostel. For how many days will the dry ration be sufficient if 5 more students join the hostel on that same day?

Solution:

(i) The relationship can be written as speed $\times$ time = distance.
(Here distance is a constant)
Therefore, if t hours are required to travel 220 km then $55t = 220$. Thus $t = 4$.

(ii) If it takes t days for 15 men to complete the task, then $15 \times t = 12 \times 10$. Thus, $t = 8$.

(iii) If t is the number of days for which the dry rations is sufficient for 15 students (Initial 10 students + 5 new students), then $15 \times t = 10 \times 21$. (Note here that the three weeks have been converted to 21 days)
Therefore, $t = 14$ days.