EN 1201 – INTRODUCTORY MATHEMATICS

Section 1: Numbers and Arithmetic Operations

1.1 Types of numbers and representation on the number line

1.1.1 Types of numbers

The set of **natural numbers** is the set {1, 2, 3, …….}. This set is also called the set of **counting numbers** or the set of **positive integers**.

The set of natural numbers together with the number 0 forms the set of whole numbers.

The set of **whole numbers** is the set {0, 1, 2, 3, …….}.

Consider the problem of subtracting the number 8 from the number 5. There is no solution to this problem which is a whole number. The problem ‘‘5 – 8’’ will have a solution if the set of whole numbers is extended to include the negative integers.

The set of **negative integers** is the set {……-4, -3, -2, -1}.

When we combine these sets together we obtain the set of integers.

The set of **integers** is the set {….. -4, -3, -2, -1, 0, 1, 2, 3, ….}. This set is denoted by the symbol $\mathbb{Z}$.

There are **fractions** (e.g. $\frac{1}{2}$) and **decimals** (e.g. 0.135) that cannot be included in the above sets. The numbers $\frac{1}{2}$ and 0.135 belong to the set of **rational numbers**. Rational numbers can be defined in terms of fractions or decimals.

(i) The set of **rational numbers** is the set $\{\frac{m}{n} \mid m, n \in \mathbb{Z} \text{ and } n \neq 0\}$.

(ii) A **rational number** is a number that can be expressed as a **terminating** or **recurring decimal**.

The set of rational numbers is denoted by $\mathbb{Q}$.

Note that every fraction can be expressed as a terminating or recurring decimal and vice versa.

Example:

(i) The fraction $\frac{1}{4}$ corresponds to the terminating decimal 0.25.

(ii) The fraction $\frac{1}{3}$ corresponds to the recurring decimal $0.\overline{3}$. The bar indicates the group of numbers that is recurring or repeating.

Note that the sets of natural numbers, whole numbers, negative integers and integers are all subsets of the set of rational numbers.

Numbers such as $\sqrt{3}, \pi$ are not rational numbers; i.e., they cannot be expressed as a fraction or a terminating or recurring decimal number. Numbers such as these are irrational numbers.

The set of **irrational numbers** is the set of non-terminating decimal expressions.

The set of irrational numbers is denoted by $\mathbb{Q}'$.

The set of **real numbers** is the set consisting of the set of rational numbers and the set of irrational numbers.

The set of real numbers is denoted by $\mathbb{R}$.

Numbers which are prefixed by a positive or negative sign are called **directed numbers**. e.g., +3, - 1.5, -$\sqrt{3}$, $+\pi$ are directed numbers.

The set of **even integers** is the set { ….. -4, -2, 0, 2, 4, ……}; i.e., the set of integers which are divisible by 2.

The set of **odd integers** is the set {….., -3, -1, 1, 3, 5, …..}; i.e., the set of integers which are not divisible by 2.

1.1.2 <u>The number line and the ordering of numbers</u>

It is impossible to list all the real numbers. But we can use a geometrical concept, a number line to indicate the real numbers. A **number line** is a straight line that has an arbitrary point which we call zero, or the origin, and by convention, all the points to the left of zero indicate negative numbers and all the points to the right of zero indicate positive numbers. (Figure 1.1)

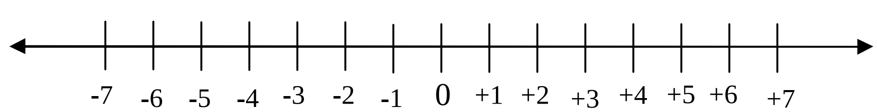

Figure 1.1

The arrows in the figure indicate that the number line extends indefinitely to the right and left of zero. Each point on the number line can be named with a real number and each real number can be represented as a point on the number line.

Between any two real numbers x and y, one of the following relationships must exist:

(i) $x = y$ (x equals y)
(ii) $x < y$ (x is less than y)
(iii) $x > y$ (x is greater than y)

The symbols < and > are called the **signs of inequality**.

If $x < y$, then x lies to the left of y on the number line. If $x > y$, then x lies to the right of y on the number line.

1.2 The basic arithmetic operations and the rules governing their application

1.2.1 The basic arithmetic operations

The basic arithmetic operations on the set of real numbers are **addition**, **subtraction**, **multiplication** and **division**. These are denoted by +, -, ×, and ÷ respectively.

The result of **adding** the two real numbers x and y is called the **sum** of x and y and is denoted by $x + y$. x and y are called the **addends** of the sum. The order in which addition is carried out is not important; $x + y = y + x$ for any two real numbers x and y.

The result of **subtracting** the real number y from the real number x (when x is greater than y) is called the **difference** between x and y and is denoted by $x – y$. The order in which we subtract numbers is very important; for example 4 – 6 is not the same as 6 – 4.

The result of **multiplying** the two real numbers x and y is called the **product** of x and y and is denoted by $x \times y$. x and y are called the **factors** of the product. The order in which we multiply numbers is not important; $x \times y = y \times x$ for any two real numbers x and y.

The result of **dividing** the real number x by the non-zero real number y is called the **quotient** of x by y and is denoted by $x \div y$. The order in which we divide numbers is very important; for example 12 ÷ 4 is not equal to 4 ÷ 12.

1.2.2 The order of operations

Numbers are often combined by a series of arithmetic operations. When this happens a definite order must be followed.

The order in which arithmetic operations are performed from left to right is as follows:
Brackets
Of
Division
Multiplication
Addition
Subtraction
Taking the first letter of each word gives the **BODMAS** rule; i.e., the contents within a bracket must be evaluated before performing any other operation (since brackets are used to indicate that the terms enclosed within them are to be considered as one quantity) and division and multiplication must be performed before addition and subtraction.

Example:

(i) $5 \times (7 - 2) = 5 \times 5 = 25$

(ii) $18 - (7 - 4) = 18 - 3 = 15$

(iii) $9 + 8 \div 2 = 9 + 4 = 13$

(iv) $3 \times 6 - 6 \div 2 + 4 = 18 - 3 + 4 = 19$

A basic set of rules for the addition and multiplication of real numbers which we call the **axioms** for the addition and multiplication of real numbers are given below. Axioms are statements that are accepted as being true without proof.

Axioms for the Addition and Multiplication of Real Numbers

Property	Addition	Multiplication
1. Closure Property	If x, y are real numbers, then $x + y$ is a real number	If x, y are real numbers, then $x \times y$ is a real number
2. Commutative Property	If x, y are real numbers, then $x + y = y + x$	If x, y are real numbers, then $x \times y = y \times x$
3. Associative Property	If x, y, z are real numbers, then $(x + y) + z = x + (y + z)$	If x, y, z are real numbers, then $(x \times y) \times z = x \times (y \times z)$
4. Identity Property	There is a unique number 0 called the **additive identity** such that $x + 0 = 0 + x = x$ for all real numbers x	There is a unique number 1 called the **multiplicative identity** such that $x \times 1 = 1 \times x = x$ for all real numbers x
5. Inverse Property	For each real number x, there is a unique real number $-x$ called the **additive inverse** of x, such that $x + (-x) = (-x) + x = 0$	For each non-zero real number x, there is a unique real number $\frac{1}{x}$ called the **multiplicative inverse** (reciprocal) of x, such that $x \times \frac{1}{x} = \frac{1}{x} \times x = 1$.
6. Distributive Property	If x, y, z are real numbers, then $x \times (y + z) = (x \times y) + (x \times z)$ and $(x + y) \times z = (x \times z) + (y \times z)$	

Note:

(i) $x - y = x + (-y)$ for real numbers x and y.

(ii) $x \div y = x \times \frac{1}{y}$ for real numbers x and y with $y \neq 0$

We sometimes write xy or $x.y$ for $x \times y$

1.2.3 The absolute value of a number

The **absolute value** of the real number x symbolized by $|x|$ is

$$|x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

We note from the definition that the absolute value of a real number cannot be negative. If we consider the number line, the absolute value of a real number is the distance to the number from the origin.

1.2.4 Rules for the addition of signed numbers

Rule I: If a and b have like signs (i.e., either both are positive or both are negative), compute $|a| + |b|$ and place the common sign of a and b in front of the sum. The result is the **sum** of a and b.

Rule II: If a and b have unlike signs (one positive and the other negative) and $|a| = |b|$, then $a + b = 0$.

Rule III: If a and b have unlike signs and $|a| \neq |b|$, subtract the smaller absolute value from the larger absolute value and place the sign of the number having the larger absolute value in front of the difference. The resulting number is the **sum** of a and b.

Example:

(i) (+4) + (+8) = + (4 + 8) = + 12
(ii) (-4) + (-8) = - (4 + 8) = -12
(iii) (-4) + (+8) = + (8 – 4) = + 4
(iv) (+5) + (-5) = +(5-5) = 0
(v) (+4) + (-8) = - (8 – 4) = -4

The operation of subtraction of real numbers in terms of the operation of addition is (as given before)

$$x - y = x + (-y)$$

We use this to find the difference of signed numbers.

Example:

(i) $(+4) - (+8) = (+4) + (-8) = -(8 - 4) = -4$
(ii) $(-4) - (+8) = (-4) + (-8) = -(4 + 8) = -12$

1.2.5 Rules for the multiplication and division of signed numbers

Rule IV: If a and b have like signs, place a plus sign in front of the product (or quotient) of their absolute values. The resulting number is $+(|a|\times|b|)$ (or $+\frac{|a|}{|b|}$)

Rule V: If a and b have unlike signs, place a negative sign in front of the product (or quotient) of their absolute values. The resulting number is $-(|a|\times|b|)$ (or $-\frac{|a|}{|b|}$)

Example:

(i) $(+6) \times (+7) = +42$
(ii) $(-6) \times (-7) = +42$
(iii) $(-6) \times (+7) = -42$
(iv) $(+6) \times (-7) = -42$
(v) $(-4) - (-8) = (-4) + (-(-8)) = (-4 + (-1\times -8)) = (-4) + (+8) = +4$

1.3 Application of basic arithmetic operations on different types of numbers

We will first consider the least common multiple and highest common factor of two or more natural numbers. This knowledge is necessary to simplify fractions.

1.3.1 Prime numbers and factors

An integer x is a **factor** of another integer y, if x divides y without remainder. Then y is said to be a **multiple** of x.

Example: 2 is a factor of 12. The other factors of 12 are 1, 3, 4, 6 and 12. Therefore, 12 is a multiple of 1, 2, 3, 4, 6 and 12.

A **prime number** is a positive integer which has no factors except 1 and itself.

Example: 2, 3, 5, 7, 11, 13, 17 and 19 are the first few prime numbers.

Note that 2 is the only even number which is a prime number. All other even numbers have 2 as a factor and hence are not prime numbers.

A factor which is a prime number is called a **prime factor**.

A **composite number** is an integer which has other factors beside itself and 1.

Example:
6 is a composite number since 2 and 3 are also factors of 6 (apart from 1 and 6).

1.3.2 Common factors and relatively prime numbers

A factor which divides two or more numbers is called a **common factor** of the numbers. Thus 2 is a common factor of both 6 and 10.

Two integers are said to be **relatively prime to each other** when their only common factor is 1.

Example:

The factors of 20 are 1, 2, 4, 5 10 and 20
The factors of 27 are 1, 3, 9 and 27.
Thus 1 is the only common factor of 20 and 27. Therefore, 20 and 27 are relatively prime.

1.3.3 Highest common factor

The **highest common factor** of two or more positive integers is the greatest integer which divides each of them exactly. The abbreviation H.C.F. is used for the word highest common factor. The term greatest common divisor (G.C.D.) is sometimes used instead of the term highest common factor.

Example:
The factors of 18 are 1, 2, 3, **6**, 9 and 18
The factors of 24 are 1, 2, 3, 4, **6**, 8, 12 and 24.
Therefore the H.C.F. of 18 and 24 is 6.

1.3.4 Least common multiple

An integer which is exactly divisible by two or more integers is called a **common multiple** of them.

Thus 12 is a common multiple of 2, 3 and 6.

The **least common multiple (L.C.M.) (**also called the lowest common multiple) of two or more positive integers is the least positive integer which contains each of the numbers as a factor.

The least common multiple of a set of numbers can be obtained by writing the numbers as products of their prime factors

<u>Example</u>:
The least common multiple of 30 and 18 is 90.
If we write 30 and 18 as products of their prime factors we obtain:
$30 = 2 \times 3 \times 5$ $\qquad\qquad$ $18 = 2 \times 3 \times 3$
Therefore L.C.M. of 30 and 18 equals $= 2 \times 3 \times 3 \times 5 = 90$

1.3.5 Fractions

When the unit is divided into a number of equal parts and one or more of these parts are considered, the result is called a **fraction**.

For example, suppose the unit is divided into 8 equal parts. Then each part is one eighth of the unit. If we consider 5 parts we get five eighths of the unit. These two fractions are written as $\frac{1}{8}$ and $\frac{5}{8}$ respectively.
Thus a fraction is expressed by two numbers, one over the other, separated by a horizontal bar. The lower number which expresses the number of equal parts into which the unit is divided is called the **denominator**. The upper number which expresses the number of parts considered is called the **numerator**.
Fractions expressed in this manner are called **vulgar** or **common fractions**.

A fraction may also be defined as the result of dividing one number (the numerator) by another (the denominator); for example $\frac{5}{8}$ can be considered as 5 divided by 8.

The value of a fraction is not altered by multiplying or dividing the numerator and denominator by the same quantity; i.e., $\frac{a}{b} = \frac{ma}{mb}$ where a, b and m are natural numbers. $\frac{a}{b}$ and $\frac{ma}{mb}$ are said to be **equivalent fractions**.

When a fraction has been so simplified that its numerator and denominator have no common factors apart from 1, it is said to be **reduced to its lowest terms**.

<u>Example</u>:
$\frac{50}{125} = \frac{25 \times 2}{25 \times 5} = \frac{2}{5}$. Thus $\frac{50}{125}$ reduced to its lowest terms is equal to $\frac{2}{5}$.

A fraction such that the numerator is less than the denominator is called a **proper fraction**. A fraction such that the numerator is greater than the denominator is called an **improper fraction**.

Example:

$\frac{3}{10}$ is a proper fraction.

$\frac{15}{10}$ is an improper fraction.

Consider the improper fraction $\frac{15}{10}$.

$$\frac{15}{10}=\frac{10+5}{10}=\frac{10}{10}+\frac{5}{10}=1+\frac{1}{2}=1\frac{1}{2}$$

As seen by the above example, any improper fraction is made up of one or more units together with a proper fraction. An improper fraction expressed in terms of units and proper fractions is called a **mixed number**.

Example:

$\frac{27}{4}=6\frac{3}{4}$. Thus the improper fraction $\frac{27}{4}$ written as a mixed number is $6\frac{3}{4}$.

To change a mixed number into an improper fraction, the whole number must be multiplied by the denominator of the fractional part and then the numerator must be added to this. This sum becomes the numerator of the improper fraction while the denominator remains the same.

Comparison of fractions

Since a fraction can be considered as the result of dividing the numerator by the denominator, when two fractions have the same denominator, the greater fraction is that which has the greater numerator. When two fractions have the same numerator, the greater is that which has the lesser denominator.

Example:

$\frac{4}{10}$ is less than $\frac{7}{10}$

$\frac{7}{10}$ is greater than $\frac{7}{12}$.

Two or more fractions with different denominators can be compared by replacing the fractions by equivalent fractions, all having the same denominator. A common denominator can be obtained by considering the L.C.M. of the denominators of all the fractions that are to be compared. This is called the lowest common denominator.

Example: Arrange the fractions $\frac{5}{7},\frac{2}{3},\frac{7}{9}$ in order of magnitude.

The denominators of the fractions are 7, 3 and 9 respectively. The L.C.M. of these three numbers is 63.

$$\frac{5}{7} = \frac{5\times 9}{7\times 9} = \frac{45}{63}, \qquad \frac{2}{3} = \frac{2\times 21}{3\times 21} = \frac{42}{63}, \qquad \frac{7}{9} = \frac{7\times 7}{9\times 7} = \frac{49}{63}.$$

Therefore the numbers in order of magnitude are $\frac{2}{3}, \frac{5}{7}, \frac{7}{9}$

Addition and subtraction of fractions

1. When adding or subtracting fractions with the same denominator, we take the sum or difference of the numerator, retaining the common denominator.
2. When adding (or subtracting) fractions with different denominators, they must be first expressed as fractions that have an identical denominator. Then the numerators should be added (or subtracted).
3. In adding a series of fractions, some of which are mixed numbers, it is convenient to add the integral and fractional parts separately. Therefore, any improper fractions should be first converted into mixed numbers.

Note: In simplifying fractions, results should be brought to their **lowest terms** and improper fractions should be expressed as **mixed numbers**.

Example:

(i) Simplify the following expressions

(a) $\frac{2}{5} + \frac{5}{6} - \frac{3}{8}$

(b) $5\frac{6}{7} + 3\frac{3}{5}$

(c) $2 - \frac{3}{8}$

Solution:

(a) $\frac{2}{5} + \frac{5}{6} - \frac{3}{8}$

The L.C.M. of the denominators 5, 6 and 8 is 120.

Therefore,

$$\frac{2}{5} + \frac{5}{6} - \frac{3}{8} = \frac{2\times 24}{5\times 24} + \frac{5\times 20}{6\times 20} - \frac{3\times 15}{8\times 15} = \frac{48}{120} + \frac{100}{120} - \frac{45}{120} = \frac{48+100-45}{120} = \frac{103}{120}$$

(b) $5\frac{6}{7} + 3\frac{3}{5} = (5 + 3) + \frac{6}{7} + \frac{3}{5} = 8 + (\frac{30}{35} + \frac{21}{35}) = 8 + \frac{51}{35} = 8 + 1\frac{16}{35} = 9\frac{16}{35}$

(c) $2 - \frac{3}{8} = \frac{16}{8} - \frac{3}{8} = \frac{13}{8} = 1\frac{5}{8}$

Multiplication and division of fractions

An expression of the form $\frac{4}{7}$ of $\frac{2}{5}$, read as 'four sevenths of two fifths' is called a **compound fraction**, while $\frac{4}{7}$ of unity is called a **simple fraction.**

When the product of two numbers is unity, each is called the **reciprocal** of the other.

Example:

$\frac{3}{5}$ and $\frac{5}{3}$ are reciprocals of each other since $\frac{3}{5}\times\frac{5}{3}=\frac{15}{15}=1$.

$6\frac{1}{3}$ and $\frac{3}{19}$ are reciprocals since $6\frac{1}{3}=\frac{19}{3}$

Rules for the multiplication and division of fractions

1. When multiplying a fraction by an integer (or an integer by a fraction), we have to only multiply the numerator by the integer.
2. When dividing a fraction by an integer, we either divide the numerator by the integer (if the numerator is a multiple of the integer) or multiply the denominator by the integer.
3. When multiplying or dividing fractions involving mixed numbers, we convert the mixed numbers into improper fractions first.
4. When multiplying a fraction by a fraction, we multiply the numerators together to form the numerator of the product and we multiply the denominators together to form the denominator of the product.
5. When dividing by a fraction we multiply by the reciprocal of the fraction.

Example:

(i) Evaluate the following

(a) $\frac{12}{7}\times\frac{5}{6}$

(b) $\frac{5}{9}\div\frac{10}{3}$

(c) $\left(4\frac{1}{4}-2\frac{1}{8}\right)\times\frac{2}{5}$

(d) $2\frac{3}{4}\div\left(1\frac{3}{8}-\frac{3}{12}\right)$

(ii) A man leaves $\frac{1}{2}$ of his property to his wife. Of the remaining property he leaves $\frac{1}{3}$ to his son, and $\frac{1}{4}$ to his daughter. The rest he gives to charity. What fraction of his total property is given to charity?

Solution:

(i) (a) $\frac{12}{7}\times\frac{5}{6} = \frac{60}{42}=\frac{10}{7}=1\frac{3}{7}$

(b) $\frac{5}{9}\div\frac{10}{3} = \frac{5}{9}\times\frac{3}{10}=\frac{15}{90}=\frac{1}{6}$

(c) $\left(4\frac{1}{4}-2\frac{1}{8}\right)\times\frac{2}{5} = \left(4-2+\left(\frac{1}{4}-\frac{1}{8}\right)\right)\times\frac{2}{5}=\left(2+\left(\frac{2}{8}-\frac{1}{8}\right)\right)\times\frac{2}{5}=2\frac{1}{8}\times\frac{2}{5}=\frac{17}{8}\times\frac{2}{5}=\frac{17}{20}$

(d) $2\frac{3}{4}\div\left(1\frac{3}{8}-\frac{3}{12}\right)=\frac{11}{4}\div\left(1+\left(\frac{9}{24}-\frac{6}{24}\right)\right)=\frac{11}{4}\div 1\frac{1}{8}=\frac{11}{4}\div\frac{9}{8}=\frac{11}{4}\times\frac{8}{9}=\frac{22}{9}$

$= 2\frac{4}{9}$

(ii) The portions that his son and daughter get of the total property are respectively $\frac{1}{3}\times\frac{1}{2}=\frac{1}{6}$ and $\frac{1}{4}\times\frac{1}{2}=\frac{1}{8}$. Thus the fraction that goes to charity is $1-\left(\frac{1}{2}+\frac{1}{6}+\frac{1}{8}\right) = 1-\left(\frac{12}{24}+\frac{4}{24}+\frac{3}{24}\right)=1-\left(\frac{19}{24}\right)=\frac{5}{24}$

1.3.6 Decimals

The numbers 10, 100, 1000, 10 000 …. are known as **powers of 10**. 10 is the first power, 100 or 10^2 is the second power etc. Similarly $\frac{1}{10},\frac{1}{100},\frac{1}{1000},.....$ are known as the **inverse powers of ten**. These may also be written as $\frac{1}{10},\frac{1}{10^2},\frac{1}{10^3},.....$

Any whole number may be expressed in terms of descending powers of 10 and units.

Example: $5611 = (5 \times 10^3) + (6 \times 10^2) + (1 \times 10) + 1$

If we now extend this system fractionally beyond the units figure, we may express tenths, hundredths, thousandths, etc. To mark the position of the units figure, we place a dot called the **decimal point** immediately after the units figure. A fraction (or a whole number followed by a fraction) thus expressed in tenths, hundredths etc., is called a **decimal**.

Example: The mixed number 6 hundreds + 3 tens + 2 units + 5 tenths + 2 hundredths may be expressed in terms of figures as 632.52. The number 632.52 is read as 'six hundred and thirty two point five two'.

Multiplication and division by powers of ten

1. When a decimal is multiplied by 10, 10^2, 10^3 etc., the decimal point is moved 1, 2, 3 etc., places to the right.
2. When a decimal is divided by 10, 10^2, 10^3 etc., the decimal point is moved 1, 2, 3 etc., places to the left.

Example:

(i) $2.306 \times 100 = 230.6$

(ii) $2.306 \div 100 = 0.02306$

Addition and subtraction of decimals

Addition and subtraction of decimals can be performed in the same manner as the addition and subtraction of integers. In the addition and subtraction of decimals, the decimals should be placed so that the units' figures are in one column and consequently the decimal point in one vertical line. By this means the tenths in the given decimals form one column, the hundredths another column etc.

Example:

(i) Find the value of the following

(a) $345.607 + 1200.0008 + 489.999$

(b) $1200.0008 - 489.999$

Solution:

(a)
$$\begin{array}{r} 345.607 \\ 1200.0008 \\ \underline{489.999} \; + \\ 2035.6068 \end{array}$$

(b)
$$\begin{array}{r} 1200.0008 \\ \underline{489.999} \; - \\ 710.0018 \end{array}$$

Multiplication of decimals

Example:

(i) $0.08 \times 113 = \frac{8}{100} \times \frac{113}{1} = \frac{8 \times 113}{100} = \frac{904}{100} = 9.04$

(ii) $2.36 \times 1.3 = \frac{236}{100} \times \frac{13}{10} = \frac{236 \times 13}{1000} = \frac{3068}{1000} = 3.068$

From the above examples we see that when we multiply decimals, the number of decimal figures in a product is always equal to the sum of the number of decimal figures in the factors.

Thus multiplication of decimals may also be carried out in the following manner.

Example:

(i) 0.08×113
$8 \times 113 = 904$. Thus $0.08 \times 113 = 9.04$
(since there are only two decimal figures in the factors, the product will also have two decimal figures)

(ii) 2.36×1.3
$236 \times 13 = 3068$. Thus $2.36 \times 1.3 = 3.068$
(since the sum of the number of decimal figures in the factors is 3 the product will also have three decimal figures)

Division of decimals

When a decimal is divided by an integral value, the division is carried out just as if the decimal represented a whole number, but by placing the decimal point in the quotient, so that it corresponds to the position of the decimal point in the dividend.

Example:

(i) $4\overline{)45.708}$ = 11.427

$$\begin{array}{r} 11.427 \\ 4\overline{)45.708} \end{array}$$

(ii)

$$\begin{array}{r} 0.00117 \\ 3\overline{)0.00351} \end{array}$$

Let us now see how we divide a decimal by a decimal by considering a few examples.

Example:

(i) $4.8 \div 0.32$.

$$\frac{4.8}{0.32} = \frac{4.8}{0.32} \times \frac{100}{100} = \frac{4.8 \times 100}{0.32 \times 100} = \frac{480}{32} = \frac{60}{4} = 15$$

(ii) $36.423 \div 0.9$

$$\frac{36.423}{0.9} = \frac{36.423}{0.9} \times \frac{10}{10} = \frac{36.423 \times 10}{0.9 \times 10} = \frac{364.23}{9} = 40.47$$

From the above examples we see that any division of a decimal by a decimal can be converted into a division of a decimal by an integral value, and the procedure described above to divide a decimal by an integral value can be used.

1.3.7 Estimation, approximation and appropriate degrees of accuracy

When a numerical problem is being solved, if you are unable to get an exact value for the answer, you must consider the **degree of accuracy** required in the answer.

Example:
If a shopkeeper sells a box of 12 pencils for Rs. 80 and a person wishes to buy just one pencil, the cost would be Rs. $\frac{80}{12}$ or Rs. $6\frac{2}{3}$. Obviously it is not possible to pay Rs. $6\frac{2}{3}$. The shopkeeper would most likely charge the person Rs. 6.70, to make sure that she does not lose on the deal. Rs. 6.70 is an **approximate value** (a value close to the actual value) of the actual cost obtained by **rounding up** the price.

Most calculators display answers up to 10 digits. In most cases this degree of accuracy is not required. Therefore when using a calculator, an approximate answer with fewer digits than indicated by the calculator could be given, up to an appropriate degree of accuracy.

One method of approximating is rounding off to a number of decimal places.

Suppose we wish to round off to the 2^{nd} decimal place (i.e. to the hundredth place). Then

(i) if the digit in the 3^{rd} decimal place is **5 or more, we round up**; i.e., we increase the digit in the second decimal place by 1.

(ii) if the digit in the 3^{rd} decimal place is **less than 5, we round down**; i.e., we leave the digit in the second decimal place as it is.

Example:

(i) Round off the following numbers to the 3^{rd} decimal place.
(a) 7.86344
(b) 23.6888
(c) 11.9999

Solution:

(a) The digit in the 4^{th} decimal place of 7.86344 is 4. Therefore we round down. Thus 7.86344 rounded off to the 3^{rd} decimal place is 7.863.
(b) The digit in the 4^{th} decimal place of 23.6888 is 8. Therefore we round up. Thus 23.6888 rounded off to the 3^{rd} decimal place is 23.689.
(c) The digit in the 4^{th} decimal place of 11.9999 is 9. Therefore we round up. Thus 11.9999 rounded off to the 3^{rd} decimal place is 12.000.

When working problems we could make mistakes in calculations, especially when we use calculators. Therefore whenever possible we should **estimate** the answer using values that are close to the actual values that we are working with, and check the accuracy of our calculation as shown in the examples below.

Example:

(i) A group of students hires two bus for a total of Rs. 48 960 to go on a tour. There are 72 students in the group. Which two numbers would you divide to get an estimate of the cost per student? What is the estimated value and the actual value?

(ii) Five students treated a group of friends to a dinner at a restaurant. The total number of students at the dinner was 14. The cost per student was Rs. 198. Give an estimate of the cost of the dinner and an estimate of how much each of the 5 students had to pay if they shared the bill equally. How much was the actual cost for each of the students?

Solution:

(i) To get an estimate for the cost we could divide Rs. 49 000 by 70. The estimate would then be Rs. 700 per student. The actual cost is Rs. 48 960 divided by 72 which is Rs. 680.

Thus in our calculation if by mistake we had obtained a solution of Rs. 68 instead of Rs. 680, by considering our estimate we would realize that there was an error in our calculation.

(ii) We could estimate the total cost of the dinner by multiplying Rs. 200 by 15 which would give us Rs. 3 000. Thus an estimate of how much each of the 5 students would have to pay is Rs. 3 000 divided by 5 which is Rs. 600. The true cost of the dinner is Rs. 2 772 and the actual amount each student would have to pay is Rs. 554.40.

1.3.8 Introduction to indices

Let a denote a real number and n denote a natural number. We denote the product $\underbrace{a \times a \times ...... \times a}_{\boldsymbol{n} \text{ times}}$ by a^n.

a^n is called the $\boldsymbol{n}^{\text{th}}$ **power of a**. a is called the **base** and n the **index** (exponent) of a^n. The second and third powers of a number are known as its square and cube respectively.

We extend this to negative indices as follows:

$a^{-n} = \dfrac{1}{a^n}$, where n is a natural number and $a \neq 0$

We define $a^0 = 1$ for all $a \neq 0$.

<u>The laws of indices</u>:

Let a and b denote real numbers and m and n denote integers. Then

(i) $a^n.a^m = a^{n+m}$

(ii) $(a^n)^m = a^{nm}$

(iii) $(ab)^n = a^n.b^n$

(iv) $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$ if $b \neq 0$

(v) $\dfrac{a^n}{a^m} = \begin{cases} a^{n-m} & \text{if } n \geq m \text{ and } a \neq 0 \\ \dfrac{1}{a^{m-n}} & \text{if } n < m \text{ and } a \neq 0 \end{cases}$

(vi) $a^0=1$ if $a \neq 0$

(vii) $a^{-n} = \dfrac{1}{a^n}$ or $\dfrac{1}{a^{-n}} = a^n$ if $a \neq 0$

1.3.9 <u>Roots and surds</u>

Let n be a natural number and a a real number. The n^{th} root of a denoted by $\sqrt[n]{a}$ is defined as follows:

<u>Case 1</u>: If n is an odd number, then the n^{th} root of a is defined to be the real number b such that $b^n = a$. In this case we write $b = \sqrt[n]{a}$.

<u>Case 2</u>: If n is an even number, then the n^{th} root of a where a is a non-negative number is defined to be the non-negative real number b such that $b^n = a$ and we write $b = \sqrt[n]{a}$.

In $\sqrt[n]{a}$, $\sqrt{\ }$ is called the **radical sign**, a the **radicand** and n the **index** of the radical.

When $n = 2$ we denote $\sqrt[n]{a}$ by $\sqrt{a}$. $\sqrt{a}$ is called the **square root** of a.

$\sqrt[3]{a}$ is called the **cube root** of a.

If m, n are both integers with n a positive number, and if $\sqrt[n]{a}$ is defined then $a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$.

<u>Properties</u>:

Let a, b be real numbers and n, m be natural numbers. Then

(i) $\sqrt[n]{a^n} = a$ if n is odd, and $\sqrt[n]{a^n} = |a|$, if n is even.

(ii) $\sqrt[n]{ab} = \sqrt[n]{a}.\sqrt[n]{b}$ if n is odd or if n is even, provided that a, b are non-negative.

(iii) $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$ if n is odd and $b \neq 0$, or if n is even, provided that a is non-negative and b is positive.

(iv) $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$, if m, n are both odd; otherwise, a must be non-negative.

A **surd** is a root of a rational number which cannot be exactly determined.

Example: $\sqrt{3}, \sqrt[3]{5}$ are surds, but $\sqrt{4}, \sqrt[3]{27}$ are not surds.

Simplifying surds

A surd is said to be simplified if it is expressed in terms of the smallest possible surd.

Example: $\sqrt{20}$ when simplified equals $2\sqrt{5}$ since $\sqrt{20} = \sqrt{4\times 5}$

Multiplying and dividing surds

Any two or more surds which are n^{th} roots (square roots or cube roots etc) may be multiplied or divided by combining them under the same root sign and then simplifying as above.

Example:

(i) $\sqrt{30}\times\sqrt{10} = \sqrt{300} = \sqrt{3\times 100} = 10\sqrt{3}$

(ii) $\dfrac{\sqrt[3]{81}}{\sqrt[3]{3}} = \sqrt[3]{\dfrac{81}{3}} = \sqrt[3]{27} = 3$

Adding and subtracting surds

Only 'like surds' may be added or subtracted. For example $2\sqrt{3}$ and $-\sqrt{3}$ are like surds but $2\sqrt{3}$ and $3\sqrt{2}$ are not like surds. To add and subtract surds they should first be written in simplified form.

Example:

(i) $2\sqrt{5}+3\sqrt{7}-3\sqrt{20}+4\sqrt{7} = 2\sqrt{5}-3(2\sqrt{5})+3\sqrt{7}+4\sqrt{7}$

$$= 2\sqrt{5}-6\sqrt{5}+7\sqrt{7} = -4\sqrt{5}+7\sqrt{7}$$

Rationalizing the denominator

If a quotient with a surd in the denominator is converted into a quotient with a rational number as the denominator, we call the process rationalizing the denominator.

We will show by two examples how denominators that include surds can be rationalized.

<u>Example</u>: Rationalize the denominator of the following

(a) $\frac{3+\sqrt{5}}{\sqrt{3}}$ (b) $\frac{5}{4-\sqrt{5}}$

<u>Solution</u>:

(a) $$\frac{3+\sqrt{5}}{\sqrt{3}} = \frac{3+\sqrt{5}}{\sqrt{3}}.\frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}(3+\sqrt{5})}{\sqrt{3}.\sqrt{3}} = \frac{3\sqrt{3}+\sqrt{15}}{3}$$

(b) $$\frac{5}{4-\sqrt{5}} = \frac{5}{4-\sqrt{5}}.\frac{4+\sqrt{5}}{4+\sqrt{5}} = \frac{5(4+\sqrt{5})}{(4-\sqrt{5})(4+\sqrt{5})} = \frac{20+5\sqrt{5}}{16-5} = \frac{20+5\sqrt{5}}{11}$$