Section 2: Basic Algebra

Algebra like arithmetic deals with numbers. Both subjects employ the fundamental operations of addition, subtraction, multiplication, division, raising to a power and taking a root. In both, the same symbols are used to indicate these operations (e.g., + is used for addition, × for multiplication etc.) and the same rules govern their use. In arithmetic we use definite numbers, and we obtain definite numerical results when we perform operations on numbers, but in algebra we are mainly concerned with general expressions and general results in which letters or other symbols are used to represent numbers not named or specified. If a letter is used to represent any number from a given set of numbers, it is called a **variable**. A constant is either a fixed number such as 5 or $\sqrt{2}$, or a letter that represents a fixed (possibly unspecified) number.

Example:
Consider the formula for the area A of a circle of radius r:

$$A = \pi r^2$$

Here, A and r are variables representing the area and the radius of a circle respectively while π is a constant.

2.1 Algebraic terminology

Any collection of numbers or letters standing for numbers (or powers or roots of these), connected by the signs +, -, ×, ÷ is called an expression. Parts of an expression separated by the signs + or – are called **terms**.

An **algebraic expression** is one of the following

(i) a constant
(ii) a variable
(iii) a combination formed by performing one or more algebraic operations (addition, subtraction, multiplication, division, raising to a power or taking a root) on non-zero constants or variables

Examples of algebraic expressions: x, c, $x+c$, $3xy+y^2+2$

A **term** is a product or quotient of one or more variables (or powers or roots of these) and constants.

Examples of terms: x, x^2y, cx, $\frac{x}{y^2}$

In a term of the form $3x$, 3 is called the **numerical coefficient** of x and x is called the **literal coefficient** of 3. When two or more terms (e.g., $-4x^2$ and $6x^2$) have the same literal coefficient, we say that they are **like terms**.

Like terms can always be combined together. This process of combining like terms is called **collecting coefficients**.

Example: $3x^2 - 4xy + 2xy - 10x^2 = 3x^2 - 10x^2 - 4xy + 2xy = -7x^2 - 2xy$

An algebraic expression which consists of only one term is called a **monomial**. A **binomial expression** is an algebraic expression consisting of two terms, and a **trinomial expression** is an algebraic expression consisting of three terms. An algebraic expression with two or more terms is called a **multinomial**.

2.2 Expansion and factorization of algebraic expressions

In algebra, as in arithmetic, the order in which operations are performed is important. Operations are performed from left to right. Brackets are used to indicate that expressions enclosed within them are to be considered as one quantity. Thus, operations within brackets are performed first. All operations of multiplication and division must be performed before those of addition and subtraction. Parenthesis ( ), brackets [ ] and braces { } are used to show the order of performing operation. The general practice is that parentheses are used first and innermost, then brackets, and finally braces.

Note:
1. When an expression within brackets is preceded by the sign +, the brackets may be removed without making any change in the expression. Conversely, any part of an expression may be enclosed within brackets and the sign + prefixed, provided the sign of every term within the brackets remains unaltered.
2. When an expression within brackets is preceded by the sign -, the brackets may be removed if the sign of every term within the brackets is changed. Conversely, any part of an expression may be enclosed within brackets and the sign – prefixed, provided the sign of every term within the brackets is changed.

2.2.1 Multiplying algebraic expressions

In order to multiply algebraic expressions, we need to apply the laws of indices of section 1.

Example:

(i) $(3xy^2)(4x^3y^3) = (3 \times 4) \times (x \times x^3) \times (y^2 \times y^3) = 12 \times x^{1+3} \times y^{2+3} = 12x^4y^5$

(ii) $(2x^3y)(-3x^{-5}y^4z) = (2 \times -3) \times (x^3 \times x^{-5}) \times (y \times y^4) \times z = -6x^{-2}y^5z$

(iii) $6x^3(2x^2 - 3xy) = (6x^3 \times 2x^2) + (6x^3 \times -3xy)$ by the distributive law
$= 12x^5 - 18x^4y$

(iv) $(x + 3)(x - 4) = (x + 3)x - (x + 3)4$ by the distributive law
$= (x \times x) + (3 \times x) - (x \times 4) - (3 \times 4)$ by the distributive law
$= x^2 + 3x - 4x - 12$
$= x^2 - x - 12$

2.2.2 Factoring algebraic expressions

In this section we will do the reverse of multiplying algebraic expressions. This reverse operation is called **factoring**.

To factor out common terms we apply the distributive property $a(b + c) = ab + ac$ in reverse.

Example:

(i) $-4xy^2 + 12x^2y = 4xy(-y) + 4xy(3x) = 4xy(-y + 3x)$
(ii) $3x^2y + 6x + 5xy + 10 = 3x(xy + 2) + 5(xy + 2) = (3x + 5)(xy + 2)$

2.2.3 Factoring of trinomials

Consider the trinomial $x^2 + bx + c$ where b and c are integers.
Suppose $x^2 + bx + c = (x + p)(x + q) = x^2 + px + qx + pq = x^2 + (p + q)x + pq$.

From the above we see that if we consider a trinomial $x^2 + bx + c$ where b and c are integers, if we can find integers p and q such that $b = p + q$ and $c = pq$, then the trinomial can be factored as
$x^2 + bx + c = (x + p)(x + q)$.

Now consider the case $ax^2 + bx + c$ (where a, b and c are integers with $a \neq 0$).
Suppose
$ax^2 + bx + c = (px + q)(rx + s) = prx^2 + (ps + qr)x + qs$.

In this case the task is to find integral factors p, r of a and q, s of c such that $ps + qr = b$.

Example:
Factor the following

(i) $x^2 + 7x + 10$
(ii) $x^2 - x - 20$
(iii) $3x^2 - x - 4$

Solution:

(i) $x^2 + 7x + 10$
Here $b = 7$ and $c = 10$. The factors 2 and 5 of 10 are such that their sum $2 + 5 = 7$.
Thus $x^2 + 7x + 10 = (x + 5)(x + 2)$
(Note: The other pairs of factors of 10 are (10, 1), (-10, -1) and (-2, -5), but the sums of these do not add up to 7)

(ii) $x^2 - x - 20$

Here b = -1 and c = -20.

The factors of -20 (and the sum of the factors) are:

20 and -1 (sum 19)

-20 and 1 (sum -19)

4 and -5 (sum -1)

-4 and 5 (sum 1)

10 and -2 (sum 8)

-10 and 2 (sum -8)

Thus the factors of -20 such that the sum of the factors equals -1 are -5 and 4.

Therefore, $x^2 - x - 20 = (x - 5)(x + 4)$

(iii) $3x^2 - x - 4 = (3x - 4)(x + 1)$

2.2.4 Difference of two squares

An expression of the form $\boldsymbol{x^2 - a^2}$ is known as **the difference of two squares** and is factored as follows:

$x^2 - a^2 = (x - a)(x + a)$

2.2.5 Other useful factorizations

(i) $x^2 + 2xy + y^2 = (x + y)^2$

(ii) $x^2 - 2xy + y^2 = (x - y)^2$

(iii) $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

(iv) $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

Example:

Factor the following:

(i) $9x^2 + 6xy + y^2$

(ii) $16x^2 - 25$

(iii) $9x^2 - 30xy + 25y^2$

(iv) $x^3 - 27$

Solution:

(i) $9x^2 + 6xy + y^2 = (3x + y)^2$ using factorization (i) above

(ii) $16x^2 - 25 = (4x - 5)(4x + 5)$ difference of two squares

(iii) $9x^2 - 30xy + 25y^2 = (3x - 5y)^2$ using factorization (ii) above

(iv) $x^3 - 27 = (x - 3)(x^2 + 3x + 9)$ using factorization (iv) above

2.2.6 Algebraic Fractions

The procedures followed in simplifying arithmetic fractions can be used to simplify algebraic fractions.

Example:
Simplify the following

(i) $\dfrac{x^2+6x+9}{3xyz}\cdot\dfrac{6xy}{x^2-9}$

(ii) $\dfrac{x+3}{x+4}\div\dfrac{xy+3y}{x^2-16}$

(iii) $\dfrac{1}{x^2-1}-\dfrac{1}{(x-1)^2}$

(iv) $\dfrac{4}{x^2+x}+\dfrac{7}{4x+4}$

Solution:

(i) $\dfrac{x^2+6x+9}{3xyz}\cdot\dfrac{6xy}{x^2-9} = \dfrac{(x+3)^2}{3xyz}\cdot\dfrac{6xy}{(x-3)(x+3)} = \dfrac{2(x+3)}{z(x-3)}$

(ii) $\dfrac{x+3}{x+4}\div\dfrac{xy+3y}{x^2-16} = \dfrac{(x+3)}{(x+4)}\cdot\dfrac{(x-4)(x+4)}{y(x+3)} = \dfrac{x-4}{y}$

(iii) $\dfrac{1}{x^2-1}-\dfrac{1}{(x-1)^2} = \dfrac{1}{(x-1)(x+1)}-\dfrac{1}{(x-1)^2} = \dfrac{(x-1)}{(x-1)^2(x+1)}-\dfrac{(x+1)}{(x-1)^2(x+1)}$

$$= \frac{x-1-(x+1)}{(x-1)^2(x+1)} = \frac{-2}{(x-1)^2(x+1)}$$

(iv) $\dfrac{4}{x^2+x}+\dfrac{7}{4x+4} = \dfrac{4}{x(x+1)}+\dfrac{7}{4(x+1)} = \dfrac{16}{4x(x+1)}+\dfrac{7x}{4x(x+1)} = \dfrac{7x+16}{4x(x+1)}$

2.3 Formulae

One of the most important applications of Algebra is the use of formulae. In every form of applied science and mathematics formulae are applied.

Formulae involve three operations:
(i) Construction (ii) Manipulation (iii) Evaluation

Examples of formulae:

(i) The formula for the perimeter P of a rectangle of length a and breadth b is $P = 2a + 2b$

(ii) The formula for the area A of a rectangle of length a and breadth b is $A = a.b$

(iii) The formula for the surface area A of a sphere of radius r is $A = 4\pi r^2$

From the above examples it can be observed that in a formula**, one quantity is expressed in terms of other quantities and the formula expresses a relationship between the quantities**.

Consider the formula for the volume V of a cylinder with base radius r and altitude h:

$$V = \pi r^2 h\ .$$

If it is required to express the altitude h of the cylinder in terms of the volume V and base radius r, we would write

$$h = \frac{V}{\pi r^2}\ .$$

When one quantity (say h in the above formula) is expressed in terms of other quantities, we call the relevant quantity the **subject of the formula**.

Thus V is the subject of the formula $V = \pi r^2 h$, and h is the subject of the formula $h = \frac{V}{\pi r^2}$.

The process of transforming one formula into another is called '**changing the subject of the formula**'. Algebraic skills are required to transform one formula into another. **Evaluation** is done by substituting given values for the unknowns.

<u>Example</u>:

(i) The time of vibration t of a simple pendulum is given by the formula $t = 2\pi\sqrt{\frac{l}{g}}$. Make l the subject of the formula.

(ii) If $v^2 = u^2 + 2fs$, make s the subject of the formula and find its value when $u = 15$, $v = 20$ and $f = 5$.

(iii) Make r the subject of the formula $V = \frac{1}{3}\pi r^2 h$ and evaluate it when $V = 66$ and $h = 7$. (Assume that $\pi = \frac{22}{7}$).

Solution:

(i) $t = 2\pi\sqrt{\frac{l}{g}}$. Therefore $\sqrt{\frac{l}{g}} = \frac{t}{2\pi}$. Thus $\frac{l}{g} = \left(\frac{t}{2\pi}\right)^2$. Hence $l = \frac{gt^2}{4\pi^2}$.

(ii) $v^2 = u^2 + 2fs$. Therefore $2fs = v^2 - u^2$. Thus $s = \frac{v^2 - u^2}{2f}$. When $u = 15$, $v = 20$ and $f = 5$, $s = \frac{400 - 225}{2 \times 5} = \frac{175}{10} = 17.5$.

(iii) $V = \frac{1}{3}\pi r^2 h$. Therefore $r^2 = \frac{3V}{\pi h}$. Thus $r = \sqrt{\frac{3V}{\pi h}}$. When $V = 66$, $h = 7$ we obtain $r = \sqrt{\frac{3 \times 66}{\frac{22}{7} \times 7}} = \sqrt{9} = 3$

2.4 Evaluation of algebraic expressions

An expression is **evaluated** by replacing the variables in the expression with given numbers and performing the indicated operations. By evaluating an expression we obtain a numerical **value** for the expression.

Example:

The value of $9x^2 + 6xy + y^2$ when $x = 1$ and $y = -1$ is $9(1)^2 + 6(1)(-1) + (-1)^2 = 9 - 6 + 1 = 4$.